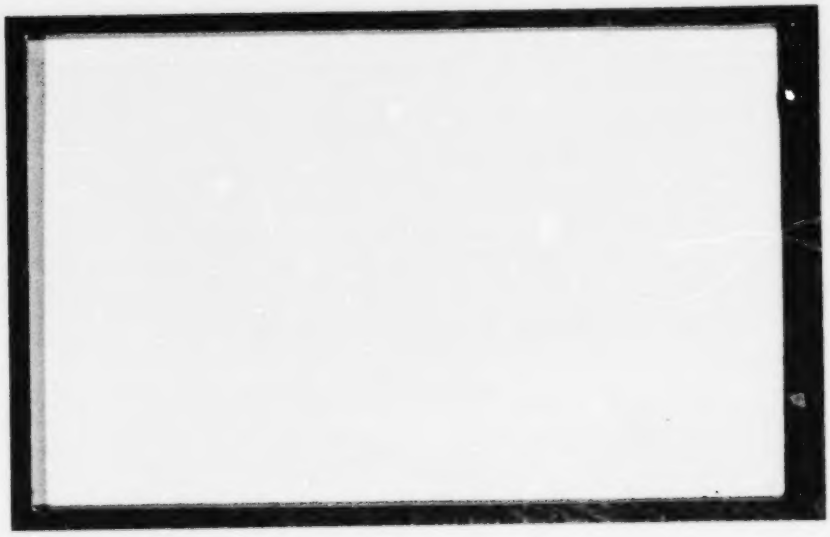
UNIVERSITÉ DE MONTRÉAL



CENTRE DE RECHERCHE SUR LES TRANSPORTS

CASE POSTALE 6128, STATION A, MONTRÉAL, QUÉBEC, H3C 3J7

SOLUTION OF THE NETWORK DESIGN PROBLEM WITH CONGESTION AND FIXED DEMANDS FOR TRAVEL BY AN ALL-OR-NOTHING TRAFFIC ASSIGNMENT ALGORITHM

by

Marc Los

This work was supported by a grant from the Transportation Development Agency of Canada.

Université de Montréal
Publication # 56 - Centre de recherche sur les transports
Janvier 1977 - Revisée mai 1977

ABSTRACT

It is shown that the network design problem with congestion and fixed demands for travel can be solved by an all-or-nothing traffic assignment algorithm, under the following assumptions: a linear investment cost function, a continuous investment variable (the practical capacity on each arc) and a volume-delay function satisfying certain assumptions. These assumptions are satisfied by the BPR function which is widely used and is a good approximation to more complex congestion functions.

Introduction

This paper is concerned with the problem of designing an optimal transportation network (for a new town for instance) when congestion is assumed on the links of the network. Although it is more realistic to assume that the real design problem is discrete[1], discrete approaches could not handle problems with several thousand links. Therefore one has to use optimization models where link capacity can vary continuously. Such a model was stated by Steenbrink [12, p. 91]. A decomposition approach was proposed by the same author to handle the problem, resulting in a master problem for which he proposed a heuristic method. Newell [11] stated an analogous model and suggested that a decomposition approach might solve it. In his paper, one can also find a qualitative discussion of square grid versus non square grid networks. Dantzig et al. [5] showed that Steenbrink's master problem could be solved exactly by traffic assignment algorithms with congestion and fixed demands such as reported by Nguyen [13] or Leblanc [7]. Abdulaal and Leblanc have proposed independently the same method as Dantzig et al. [1]. Dantzig, Maier and Landsdowne stated clearly that this approach incorporates the following features [5]: "continuous investment decision variables; system optimal traffic assignment; the total travel time on a link is expressed as a convex function of link flow and investment; and a convex investment cost function". In this paper it will be shown that, if we restrict the preceding features by imposing that the investment cost function be linear and that the volume-delay function satisfy some assumptions which are met in particular by the BPR volume-delay function the network design problem thus formulated reduces to an all-or-nothing traffic assignment problem. The sequel of the paper is

---

1 Cf references [2], [3], [6], [8] and [9] for discrete approaches to network design.

organized as follows: in section 1 we state the model and its assumptions. In section 2 we present the method of solution and in section 3 we discuss the relevance of the assumptions and then we conclude the paper.

1. Mathematical formulation and assumptions

The model

We are given a network containing a set A of potential links a. A subset of the nodes of this network are origins or destinations. The demand for travel between origin p and destination q is f_{pq}; f_{pq} subdivides among path flows $h_{k,pq}$, where k is a path between p and q; v_a is the flow on link a and z_a is the practical capacity on link a; z_a is a decision variable. The travel time on link a is a function $s_a(v_a, z_a)$ of the link flow and the link capacity; ℓ_a is the investment cost per unit of practical capacity on link a and λ is a known positive multiplier transforming dollars into time units[1]. The network design problem may then be stated as the following minimization problem.

$$\min_{h_{k,pq},\, z_a} \quad F = \sum_{a \in A} v_a\, s_a(v_a, z_a) + \sum_{a \in A} \lambda\, \ell_a\, z_a \tag{1}$$

$$\text{s.t.} \quad -f_{pq} + \sum_k h_{k,pq} = 0 \quad \forall p \quad \forall q \tag{2}$$

$$v_a = \sum_p \sum_q \sum_k \delta_{ak,pq}\, h_{k,pq} \qquad \forall a \qquad \text{where} \tag{3}$$

---

1 If we are given a budget constraint, then by a Lagrangian multiplier technique we can introduce the investment cost in the objective function and solve a series of problems for different values of λ. (Cf.refs 1 and 5).

$$\delta_{ak,pq} = \begin{cases} 1 & \text{if link a belongs to path k between p and q} \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

$$h_{k,pq} \geq 0 \quad (5)$$

$$z_a \geq 0 \quad (6)$$

Assumptions:

It is clear from the above mathematical formulation that:

H1: The decision variables z_a vary continuously.

H2: For a given network the flows are given by a system-optimal pattern.

H3: The investment cost function is linear.

Some assumptions are made on the volume-delay function which do not appear in the mathematical formulation.

A1: For each fixed $v_a \geq 0$, $s_a(v_a, z_a)$ is convex in z_a for $z_a \geq 0$. For each fixed $z_a \geq 0$, $s_a(v_a, z_a)$ is convex in v_a for $v_a \geq 0$.

A2: There exists a function σ_a such that:

$s_a(v_a, z_a) = \sigma_a(u_a)$ with $u_a = \frac{v_a}{z_a}$ for $z_a > 0$, $v_a \geq 0$

with $\sigma_a(u_a)$ convex and strictly increasing for $u_a > 0$

$s_a(0,0) = \sigma_a(\bar{u}_a)$

where $\bar{u}_a$ is the [1] solution to the equation $u_a^2 \frac{d\sigma_a}{du_a} = \lambda \ell_a$

---

1 It will be proved later that there is one and only one solution to this equation.

The function defined below can be shown to satisfy assumptions A1 and A2 (cf Appendix). It coincides with the BPR function for $z_a > 0$ and $v_a \geq 0$.

$$s_a(v_a, z_a) = t_a \left[1 + r \left(\frac{v_a}{z_a}\right)^m\right] \quad \text{for } z_a > 0 \text{ and } v_a \geq 0 \tag{7}$$

$$s_a(v_a, z_a) = d_a \quad \text{for } z_a = 0 \text{ and } v_a = 0 \tag{8}$$

$$\text{with } d_a = t_a + (\lambda \ell_a)^{\frac{m}{m+1}} (r t_a)^{\frac{1}{m+1}} m^{-\frac{m}{m+1}} \tag{9}$$

In (7) and (9) t_a is the free-flow travel time, which is assumed constant and r and m are known positive constants, with $m \geq 1$

2. Solution method

Let us prove first several results.

Theorem 1: The solution $\bar{u}_a$ to the equation

$$u_a^2 \frac{d\sigma_a}{du_a} = \lambda \ell_a \tag{10}$$

exists and is unique.

Proof: From A2 we know that $\frac{d\sigma_a}{du_a} > 0$ for $u_a > 0$ and $\frac{d^2\sigma_a}{du_a^2} \geq 0$ for $u_a > 0$

The derivative of $u_a^2 \frac{d\sigma_a}{du_a}$ with respect to u_a is $u_a^2 \frac{d^2\sigma_a}{du_a^2} + 2 u_a \frac{d\sigma_a}{du_a}$ and is always strictly positive for $u_a > 0$. Therefore $u_a^2 \frac{d\sigma_a}{du_a}$ is a strictly monotone

increasing function for $u_a > 0$, such that $u_a^2 \frac{d\sigma_a}{du_a} = 0$ for $u_a = 0$ and

$$\lim_{u_a \to \infty} (u_a^2 \frac{d\sigma_a}{du_a}) = \infty$$

Therefore $u_a^2 \frac{d\sigma_a}{du_a}$ crosses the line $\lambda \ell_a$ once and only once, for all positive values of λ.

Corrolary: $s_a(0,0) = \sigma_a(\bar{u}_a)$ is uniquely defined and by convention we set

$$\lim_{\substack{v_a \to 0+ \\ z_a \to 0+}} \frac{v_a}{z_a} = \bar{u}_a \qquad (11)$$

Theorem 2: For a solution to be optimal we must have, for all links a, either $v_a > 0$ and $z_a > 0$ or $v_a = 0$ and $z_a = 0$

Proof: We cannot have $z_a > 0$ and $v_a = 0$ at the optimum because the objective function can be diminished simply by setting $z_a = 0$ for all links a such that $v_a = 0$. And we cannot have $v_a > 0$ and $z_a = 0$ at the optimum because, according to A2 $\lim_{z_a \to 0+} s_a(v_a, z_a) = \lim_{u_a \to \infty} \sigma_a(u_a) = \infty$

therefore F would have an infinite value.

In order to solve the problems (1) - (6) we will neglect constraint (6). This is justified by the fact that the optimal solution to the problems (1) -

(5) will automatically satisfy constraint (6). This constraint is therefore redundant. Then z_a disappears completely from the constraint set. In addition, from assumption A1, and from the fact that the sum of convex functions is convex we know that a necessary and sufficient condition for z_a to be optimal is that it satisfy the equation:

$$\frac{\partial F}{\partial z_a} = - \left(\frac{v_a}{z_a}\right)^2 \frac{d\sigma_a}{du_a} + \lambda \ell_a = 0 \quad \forall_a \tag{12}$$

$$\text{or} \qquad u_a^2 \frac{d\sigma_a}{du_a} = \lambda \ell_a \qquad \forall_a \tag{10}$$

From theorem 1, we know that (10) has a unique solution $\bar{u}_a$. Since, at the optimum, we have either $v_a > 0$ and $z_a > 0$ or $v_a = 0$ and $z_a = 0$ and since we made the convention that

$$\lim_{\substack{z_a \to 0+ \\ v_a \to 0+}} \frac{v_a}{z_a} = \bar{u}_a$$

we can write that the following relationships

$$v_a = \bar{u}_a z_a \tag{13}$$

$$s_a (v_a, z_a) = \sigma_a (\bar{u}_a) \tag{14}$$

hold at the optimum.

For the BPR function, equation (10) becomes:

$$m t_a r \, v_a^{m+1} z_a^{-m-1} = \lambda \ell_a \tag{15}$$

and therefore:

$$\left(\frac{v_a}{z_a}\right)^{m+1} = \frac{\lambda \ell_a}{m t_a r} \tag{16}$$

or

$$\bar{u}_a = \left(\frac{\lambda \ell_a}{m t_a r}\right)^{\frac{1}{m+1}} \tag{17}$$

$$d_a = \sigma_a(\bar{u}_a) = t_a + r t_a \bar{u}_a^{\,m} \tag{18}$$

or

$$d_a = t_a + (\lambda \ell_a)^{\frac{m}{m+1}} (r t_a)^{\frac{1}{m+1}} m^{-\frac{m}{m+1}} \tag{19}$$

With the values of the travel times given by (19) or more generally by (14), the objective function of the master problem in $h_{k,pq}$ is:

$$F = \sum_a v_a \sigma_a (\bar{u}_a) + \sum_a \lambda \ell_a \bar{u}_a^{-1} v_a \tag{20}$$

The master problem can be stated as:

$$\underset{h_{k,pq}}{\text{Min}} \quad F = \sum_a \theta_a v_a \tag{21}$$

$$\text{s.t.} \quad f_{pq} = \sum_k h_{k,pq} \qquad \forall p,q \tag{22}$$

$$v_a = \sum_p \sum_q \sum_k \delta_{ak,pq} h_{k,pq} \qquad \forall a \tag{23}$$

$$h_{k,pq} \geq 0 \tag{24}$$

where

$$\theta_a = \sigma_a (\bar{u}_a) + \lambda \ell_a \bar{u}_a^{-1} \tag{25}$$

This is a traffic assignment problem with constant arc costs which can be solved very efficiently by many shortest path algorithms[1]. Once the shortest paths (in the sense of the θ_a - costs) between origin-destination pairs are known, the optimal v_a's are given by

$$v_a = \sum_p \sum_q \delta_{a,pq} f_{pq} \tag{26}$$

where $$\delta_{a,pq} = \begin{cases} 1 & \text{if a is on a shortest path between p and q} \\ 0 & \text{otherwise} \end{cases} \tag{27}$$

and the capacities are given by

$$z_a = \bar{u}_a^{-1} v_a \tag{28}$$

Consequently the optimal capacities will be non negative since the link flows v_a will be non negative. This solves the network design problem as represented by relationships (1) to (6).

3. Discussion of the assumptions of the model and conclusions

We will discuss in turn the relevance of each of the assumptions of the model. Hypothesis H1 (link capacities are continuous) is only a first approximation: it is made for computational purposes only. For a problem with a few hundred links, a heuristic technique to solve a discrete formulation would probably be more realistic. Hypothesis H2 is also only an approximation. If the real traffic assignment were a user equilibrium, the system-optimal formulation could be used to find lower and upper bounds to the optimum. The lower bound is simply the value of the optimum of the network design problem with system-optimal traffic assignment, while the upper bound is obtained by computing

---

1 Cf reference [14].

the travel cost and the construction cost of the network giving rise to the preceding lower bound, but assuming a user-equilibrium[1]. Assumption H3, that the investment cost is a linear function of the capacity on a link, is a compromise between the assumption of convexity and the assumption of concavity: according to Steenbrink who applied his heuristic approach to the Dutch Road Network, the investment cost function is piecewise linear and is concave first and then convex as is depicted below (cf. 12, p. 219).

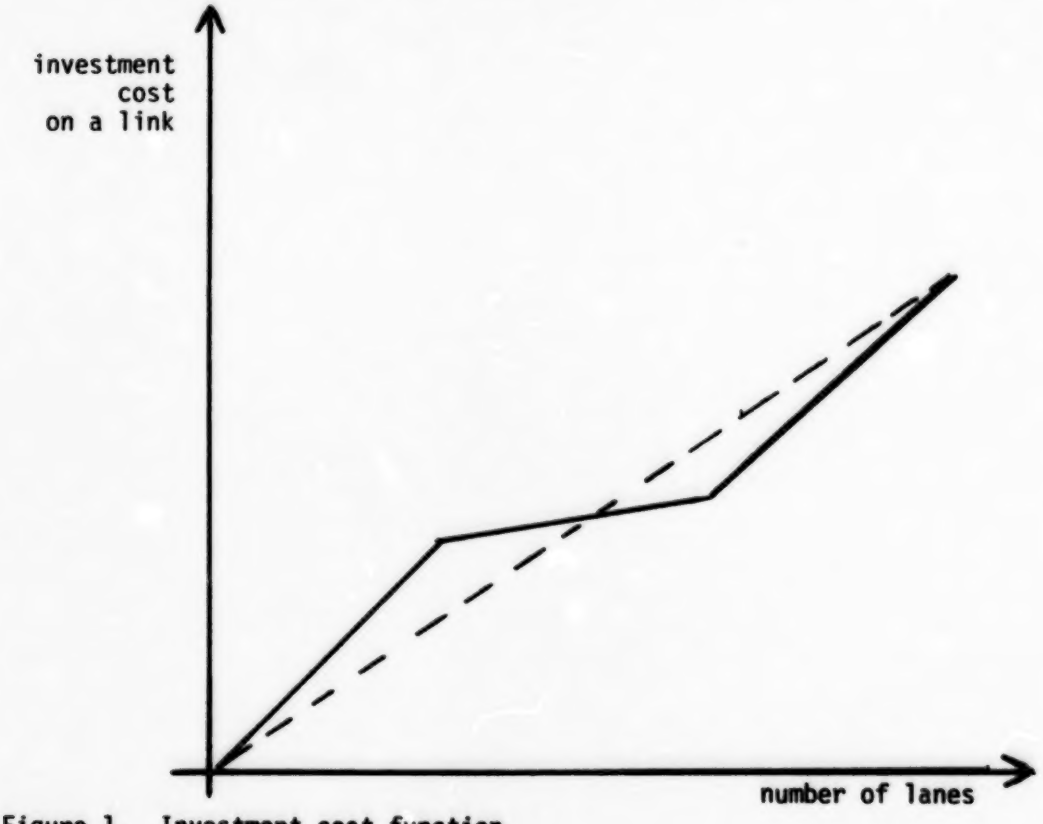


Figure 1. Investment cost function

A possible linear approximation might be the dotted line.

If the investment cost function were concave as suggested by G. Newell [11] then a linear approximation could be used to construct a lower bound and an upper bound to the optimum.

1 These two bounds are mentioned by Dantzig et al. [5].

Let ϕ be the true objective function (i.e. with a concave investment cost function), ϕ^* be the optimum of the true optimization problem. Let $(\bar{z}_a)$ be the optimal capacities obtained by solving the problem with the linear approximation F to the objective function. Then we have

$F(\bar{z}_a) \leq \phi^* \leq \phi\ (\bar{z}_a)$

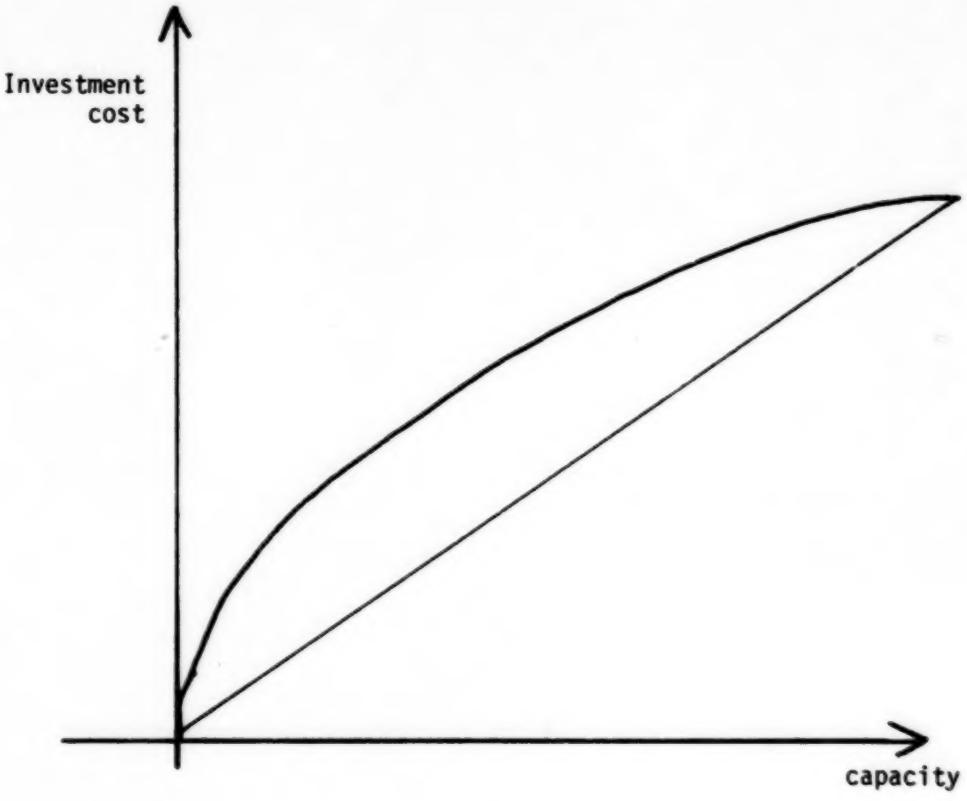


Figure 2. The linear approximation is assumed to lie below the true function for all values that can be encountered in practice.

Of course if the investment function were convex the method suggested by LeBlanc and by Dantzig et al. would be appropriate.

Concerning the assumptions A1 to A3, the fact that they are met by the BPR

function shows that they are quite realistic, since it has been shown that the BPR function enables a good approximation to more complex congestion functions. [4]

Some of the approximations made to solve the network design problem can be justified on computational grounds. However since this paper showed that the network design problem with congestion could be solved with the use of an all-or-nothing traffic assignment algorithm, the size of the problems that can be solved is limited only by the size of the networks for which shortest parth codes can be applied. [14]

ACKNOWLEDGEMENTS

The author is grateful to Michael Florian, Sang Nguyen, Patrice Marcotte and anonymous referees for their comments on some earlier drafts of this article.

REFERENCES

1. Abdulaal, M., L. LeBlanc, "An Efficient Dual Approach to the Urban Network Design Problem" (processed papers), 1976.

2. Billheimer, J.W. and P. Gray, "Network Design with Fixed and Variable Cost Elements". Transportation Science (1973), 49-74.

3. Boyce, D.E., A. Farhi, R. Weischedel, "Optimal Network Problem: A Branch and Bound Algorithm" Environment and Planning 5 (1973), 519-33.

4. Branston, D., "Link Capacity Functions: A Review". Transportation Science, Vol. 10, Nº 4, pp. 223-236.

5. Dantzig, G.B., S.F. Maier and Z.F. Lansdowne (Control Analysis Corporation) "An Application of Decomposition to Transportation Network Analysis" Report Nº DOT-1059-1 March 1976.

6. Dionne, R., M. Florian, "Exact and Approximate Algorithms for Optimal Network Design". Publication # 41, Centre de Recherche sur les Transports, Université de Montréal, February 1977 (processed paper).

7. LeBlanc, L.J., Mathematical Programming Algorithms for Large Scale Network Equilibrium and Network Design Problems". Transportation Research, 1975

8. LeBlanc, L.J., "An Algorithm for the Discrete Network Design Problem" Transportation Science, 9 # 3, 1975.

9. Los, M., "An Optimal Network Problem Without Congestion: Some Computational Results". Publication # 40, Centre de recherche sur les Transports, Université de Montréal, November 1976 (processed paper).

10. Morlok, E.K., J.L. Schofer, W.P. Pierskalla, R.E. Marsten, S.K. Agarwal, J.L. Edwards, L.J. LeBlanc, D.T. Spacek and J.W. Stoner, "Development and Application of a Highway Network Design Model" Department of Civil Engineering, Northwestern University, 1973.

11. G.F. Newell, "Optimal Network Geometry". Proceedings of the 6th International Symposium on Transportation and Traffic Flow Theory, pp. 561-580.

12. Steenbrink, P.A., "Optimization of Transport Networks". John Wiley, 1974.

13. Nguyen, S. "An Algorithm for the Traffic Assignment Problem". Transportation Science, Vol. 8, 1974, pp. 203-216.

14. Dial, R., Glover, F., Karney, D., Klingman, D.: A computational analysis of alternative algorithms and labeling techniques for finding shortest path trees. Research Report CCS 291. Center for Cybernetic Studies. The University of Texas, Austin, Texas.

APPENDIX

Proof that the BPR function satisfies A1 to A2:

A1: $s_a(v_a, z_a) = t_a + t_a\, r\, \frac{v_a^m}{z_a^m}$ with $m \geq 1$, $t_a > 0$, $r > 0$.

$$\frac{\partial^2 s_a}{\partial z_a^2} = m(m+1)\, t_a\, r\, v_a^m\, z_a^{-m-2} \geq 0$$

for $v_a \geq 0$ and $z_a > 0$

$$\frac{\partial^2 s_a}{\partial v_a^2} = m(m-1)\, \frac{t_a r}{z_a^m}\, v_a^{m-2} \geq 0$$

for $v_a \geq 0$ and $z_a > 0$

A2: $s_a(v_a, z_a) = t_a + t_a\, r\, u_a^m$

with $u_a = \frac{v_a}{z_a}$

$$\frac{d\sigma_a}{du_a} = m\, t_a\, r\, u_a^{m-1} > 0 \quad \text{for } u_a > 0$$

$$\frac{d^2\sigma_a}{du_a^2} = m(m-1)\, t_a\, r\, u_a^{m-2} \geq 0 \quad \text{for } u_a > 0.$$